THE

# MATHEMATICAL GAZETTE.

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## OUR LOCAL BRANCH.

THE usual meeting of the North Wales branch was held at Dr. Bryan's house, Bangor, on Nov. 21. Mr. Madoc Jones, head-master of Beaumaris County School, read an interesting paper on the teaching of algebra in schools, in which he expressed a strong opinion that algebra should not be started too early, the transition from arithmetic being made almost imperceptibly; that the solving of equations should be introduced as soon as possible, and applied to simple problems; and that the principles underlying algebraic methods and problems should be insisted upon, mere mechanical proficiency not being regarded as the chief end. The difficulty of making the "rules of signs" intelligible was also discussed, and graphical illustrations were recommended.

Dr. Bryan urged the early introduction of logarithms, pointing out the mistake of postponing the subject until indices had been mastered, partly because logarithms had been invented without reference to an incommensurable base.

Then followed a discussion about examinations, complaint being made of the difficulty in Wales of satisfying two or sometimes three different bodies of examiners, of the want of sympathy between teachers and examiners, and of the consequent absence of scope for individual work.

The President and Secretary were re-elected for 1909, and it was agreed to hold the next meeting, on Feb. 13, at Bangor.

## THE DIVISION OF THE CIRCLE.

THE possibility of solving the problem of the elementary division of a circumference into equal parts was known to the ancients for the cases

$$n = 2^p,\ n = 3,\ n = 5, \text{ and their products.}$$

Gauss in his *Disquisitiones Arithmeticae* proved the possibility of solving the problem by quadratics, and therefore geometrically with ruler and the compass, for all the prime numbers of the series

$$n = 2^{2^p} + 1.$$

The prime numbers $n$ of this series now known are 5, including the values $n=3$, $n=5$, for

$$\begin{aligned} p&=0, \quad & n&=3; \\ p&=1, \quad & n&=5; \\ p&=2, \quad & n&=17; \\ p&=3, \quad & n&=257; \\ p&=4, \quad & n&=65537. \end{aligned}$$

The case of $n=17$, investigated by Gauss himself with the solution of $x^{17}=1$, was geometrically interpreted by Serret in his *Algèbre Supérieure* by the geometrical representation of the roots of the quadratic equations obtained with the help of ruler and compass. The demonstration as given by Serret was modified by Bachmann and reproduced in his book, *Die Lehre von der Kreisteilung* (1872).

Another geometrical interpretation was given by von Staudt (*Crelle*, 1842) by the help of the ruler and a fixed circle in the plane of the figure; more recently a geometrical construction of the roots solving the problem by means of the compass alone was given by Gérard (*Math. Annalen*, 1897).

The case $n=257$, investigated by Richelot in his work, *De resolutione algebricâ aequationis* $x^{257}=1$, *sive de divisione circuli per bisectionem anguli septies repetitam in partes* 257 *inter se aequales commentatio coronata*, received geometrical interpretation at the hands of Affolter and Pascal (*Rendiconto della R. Acc. di Napoli*, 1887).

Case $n=65537$ was investigated by Prof. Hermes of Lingen (*Göttinger Nachrichten*, No. 3, 1894): the papers containing his very laborious researches, which took up ten years of his life, are preserved in the mathematical seminary at Gottingen.

Besides the cases already mentioned, others also exist in which the problem of the division of the circle can be solved by quadratics: these cases belong to the series

$$n=2^p,\ (2^{p_1}+1),\ (2^{p_2}+1),\ldots(2^{p_s}+1),$$

$p,\ p_1,\ p_2,\ldots p_s$ being different.

These cases were dealt with by Prof. Enriquez of the University of Bologna in his brochure entitled *Sulle equazioni algebriche risolubili per radicali quadratici e sulla costruibilita dei poligoni regolari.*

In all other cases solution geometrically by means of the rule and compass or algebraically by means of quadratic is impossible.

Thus, *e.g.* for $n$ a prime number not comprised in the Gaussian series, writing $n-1$ under the form of the product of its prime factors,

$$n-1=2^\alpha 3^\beta 5^\gamma \ldots p^\delta,$$

it is proved that the problem is reducible to the solution of

| | | | | |
|---|---|---|---|---|
| $\alpha$ | equations | of the | 2nd | degree. |
| $\beta$ | ,, | ,, | 3rd | ,, |
| $\gamma$ | ,, | ,, | 5th | ,, |
| $\delta$ | ,, | ,, | $p$th | ,, |

It follows that for the geometrical construction of the roots of such equations, instruments more complicated than the rule and compass are necessary, and such instruments should be used for drawing the algebraic curves solving the said equations.

The problems of the 'trisection of an angle,' the 'duplication of the cube' (the Delian problem), and the 'squaring of the circle,' tyrannised, as Prof. Gino Loria has it, all the geometry of the ancients. We may remind the reader that many solutions of that famous problem are known, such as those founded on the conic sections, the 'conchoid of Nicomedes,' the 'Pascal's snail,' besides those given by various instruments for trisection based generally speaking on the metrical properties of the aforesaid curves. Klein, referring to constructions for the polysection of angle or circle, states that the practical execution of all such constructions by means of higher algebraic curves remains unsolved.

Hereafter is shown the solution of the problem of polysection of any angle and the circle, for any whole value of $n$, by means of algebraic curves that can be graphically obtained.

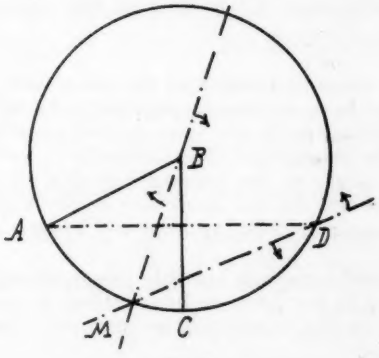


FIG. 1.

"Given a circle and an angle at its centre, to find a family of curves which shall determine the $n$-section of the angle and shall divide the circle according to the vertices of the regular inscribed $n$-gon, $n$ being any whole number."

This is equivalent to proving that these curves must give all the roots of the problem of the $n$-section of the angle.

Let $ABC$ be any angle at the centre of the circle ($B$). Let a point $M$ on the circumference divide the arc $AC$, which measures the given angle, in the ratio $\frac{AC}{MC}=n$.

Let $M$ be conceived as the intersection of $BM$ with $DM$ ($D$ being the image of $A$ in $BC$), and supposing $BM$ to have rotated through the angle $ADM$ measured by

$$\tfrac{1}{2}AM=\tfrac{1}{2}\frac{n-1}{n}.CA.$$

The ratio of the two arcs is the ratio of velocity of rotation of $BM$ and $DM$, as shown by the arrows in Fig. 1, which will result in their crossing on the circle in the point $M$ cutting off the $n$-section of the angle; this ratio is a definite function of $n$ expressed by

$$K=\frac{\frac{1}{2}AM}{\frac{AC}{n}}=\frac{n-1}{2}.$$

The problem of $n$-section of the angle is thus kinematically analogous to the rotation of two straight lines with a given ratio of velocity, being a simple function of $n$.

The order of the curve-locus of intersections of the two straight lines is also, as will be proved, a definite function of $n$.

For $n=3$, the case of trisection of the angle, the ratio of velocity will be $\frac{n-1}{2}=1$.

Noting the sense of rotation of the generating straight lines we see that we have as locus an equilateral hyperbola, since the kinematical definition is the same as the projective definition of two pencils of straight lines inversely congruous and not perspective. However, the kinematical view gives in addition a variety of curves for the trisection of the angle—as for the $n$-section—essentially differing from the known curves already mentioned.

The equilateral hyperbola not only passes through the poles of rotation $B$ and $D$, but gives as intersections on the circle points corresponding to the three roots of the cubic equation defining the problem, *i.e.*

$$\frac{ABC}{3},\ \frac{ABC+2\pi}{3},\ \frac{ABC+4\pi}{3}.$$

This property of giving all the roots of the problem is common to the loci defined by the intersection of the straight lines rotating as explained.

We must, however, distinguish two separate cases corresponding to $n=2p+1, \quad n=2p.$

Let $n$ be an odd number.

The ratio $k=\frac{n-1}{2}=p$ is an integer: that is to say the two generating straight lines $BM$ and $DM$ return to their initial mutual position after a semi-rotation of $BM$; in the successive semi-rotation the locus is repeated.

The intersections of the curve with the circle, in addition to that corresponding to the point $D$, are $n$ in number and are distributed in the vertices of the regular inscribed $n$-gon, one vertex being at $M$.

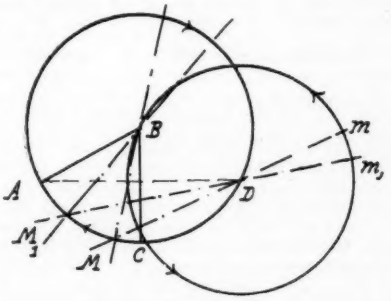


FIG. 2.

Let us then consider a point $M_1$ on the circumference such that the arc $MM_1=\frac{2\pi}{n}$; it will suffice to show that this point must belong to the locus. In fact, the ratio of the angle $MDm_1$ measured in the sense of its generation, to the corresponding angle $M_1BM$ is $\frac{n-1}{2}$, as follows from the value of $MDm_1$, the supplementary angle to $M_1DM$ measured by $\frac{1}{2}M_1M=\frac{\pi}{n}$, is

$$\frac{n-1}{n}\pi \text{ or } \frac{n-1}{2} \text{ times } \frac{2\pi}{n}=M_1BM. \qquad \text{Q.E.D.}$$

The same demonstration can be applied to the other $n-2$ vertices.

CAMILLO MANZITTI.

*(To be continued.)*

## NOTE ON THE GROUPS OF SUBTRACTION AND DIVISION, AND ON THE HYPERBOLIC FUNCTIONS.

THE objects of the present note are to exhibit an interesting property of some groups of subtraction and division, and to present a method of using such groups in the study of some elementary properties of the hyperbolic

functions. The four finite groups of subtraction and division which transform every rational number into a rational number are of orders 4, 6, 8, and 12 respectively.* With respect to each of these groups all the numbers, with the exception of two more than the order of the group, are divided into sets of conjugates, each set containing as many distinct numbers as the order of the group. The property to which we referred in the first sentence is that each of these sets contains two and only two negative numbers whenever the set is composed of real numbers and the number from which we subtract is positive. In the special case, when the symmetric group of order 6 is obtained by subtracting from unity and by dividing unity, this theorem is clearly illustrated by the known fact that all such sets of six conjugate numbers may be obtained by assigning successively different values to $x$, $0<x<\frac{\pi}{4}$, in the following functions:

$$\sin^2x,\ \sec^2x,\ -\cot^2x,\ \cos^2x,\ \csc^2x,\ -\tan^2x.$$

Thus if $\lambda=\sin^2x$, the six values found by operating upon $\lambda$ with the operations of the group are:

$$\lambda=\sin^2x,\quad 1-\lambda=\cos^2x,\qquad \frac{1}{\lambda}=\csc^2x,$$

$$\frac{1}{1-\lambda}=\sec^2x,\quad \frac{\lambda}{\lambda-1}=-\tan^2x,\quad \frac{\lambda-1}{\lambda}=-\cot^2x.$$

Hence one property of this group, which is also presented by the six anharmonic ratios of four points, is to associate four positive numbers with two negative ones, and *vice versa*. That this property is preserved when instead of subtracting from unity and dividing unity we subtract from $x_1$ and divide $x_2$, $x_1$ and $x_2$ being any two positive numbers satisfying the condition $x_1^2=x_2$, follows directly from the fact that the resulting six operations may be represented by

$$n,\quad \frac{x_1^2}{x_1-n},\quad x_1-\frac{x_1^2}{n},\quad x_1-n,\quad \frac{x_1^2}{n},\quad \frac{x_1n}{n-x_1}.$$

Two, and only two, of these numbers are always negative when $n$ is a positive number.

From this point of view we may regard the six anharmonic ratios of four points as constituting a very special case of six related distinct numbers which always include four positive and two negative ones. In the case of the first of the four groups mentioned above, viz. the group of order 4 generated by the operations of subtracting from 0 and dividing an arbitrary number, it is clear that two of the four distinct conjugates are always positive, while the other two are negative. Moreover, it is not difficult to verify that only two negative numbers occur in each complete set of distinct conjugates under each of the remaining two groups in question wherever $x_1$ is positive, since the operations of these groups may be represented as follows:

$$n,\ x_1-n,\ \frac{x_1^2}{2n},\ \frac{x_1^2}{2x_1-2n},\ \frac{2x_1n-x_1^2}{2n},\ \frac{x_1^2-2x_1n}{2x_1-2n},\ \frac{x_1^2-x_1n}{x_1-2n},\ \frac{x_1n}{2n-x_1};$$

$$n,\ x_1-n,\ \frac{x_1^2}{3n},\ \frac{x_1^2}{3x_1-3n},\ \frac{3x_1n-x_1^2}{3n},\ \frac{x_1n}{3n-x_1},\ \frac{2x_1n-x_1^2}{3n-x_1},\ \frac{3x_1n-x_1^2}{6n-3x_1},$$

$$\frac{3x_1n-2x_1^2}{6n-3x_1},\ \frac{2x_1n-x_1^2}{3n-2x_1},\ \frac{x_1n-x_1^2}{3n-2x_1},\ \frac{2x_1^2-3x_1n}{3x_1-3n}.$$

When $x_1$ and $n$ are both negative, $x_1-n=-(|x_1|-|n|)$, and hence it results that a change in the sign of $x_1$ without changing $x_2$ changes the signs

* *Quarterly Journal of Mathematics*, vol. 37 (1905), p. 80.

of all the numbers of a system of conjugates under the group generated by $x_1 - n$ and $\frac{x_2}{n}$, but does not affect their absolute values. From this it follows that a system of distinct conjugates, or a non-degenerate system, under one of the groups in question, contains only two positive numbers when $x_1$ is negative. For example, the numbers which are conjugate under the group generated by subtracting from $x_1 = -1$ and by dividing $x_2 = 1$ may be obtained by assigning successively different values to $x$, $0 < x < \frac{\pi}{4}$, in the following six functions:

$$-\sin^2 x, \quad -\sec^2 x, \quad \cot^2 x, \quad -\cos^2 x, \quad -\csc^2 x, \quad \tan^2 x.$$

The only set of two numbers which forms a complete set of conjugates under this group is the pair of imaginary cube roots of unity, and the only two sets of conjugates consisting of three numbers only are

$$1, \quad -2, \quad -\tfrac{1}{2} \quad \text{and} \quad 0, \quad -1, \quad \infty.$$

All other numbers are transformed into six distinct conjugates by the operations of this group, two of these conjugates being positive, while the remaining four are negative. The more general result which has been proved may be stated as follows: *Any complete non-degenerate set of conjugates under one of the four finite groups of subtraction and division which transform every rational number into a rational number, contains either only two negative numbers or only two positive numbers as the number from which we subtract is positive or negative.*

From the well-known equations

$$1 - \operatorname{sech}^2 x = \tanh^2 x, \quad \frac{1}{\operatorname{sech}^2 x} = \cosh^2 x,$$

it follows that the squares of the hyperbolic functions may be obtained from any one of them by means of the group generated by the operations $(s_1, s_2)$ of subtracting from unity and dividing unity. The (1, 1) correspondence between the operations of obtaining the hyperbolic and the trigonometric functions from one of them may perhaps become clearer from the following array:

| | | | | | |
|---|---|---|---|---|---|
| $1$ | $s_1 s_2$ | $s_2 s_1$ | $s_1$ | $s_2$ | $s_1 s_2 s_1$ |
| $\operatorname{sech}^2 x$ | $\coth^2 x$ | $-\sinh^2 x$ | $\tanh^2 x$ | $\cosh^2 x$ | $-\operatorname{csch}^2 x$ |
| $\sin^2 x$ | $\sec^2 x$ | $-\cot^2 x$ | $\cos^2 x$ | $\csc^2 x$ | $-\tan^2 x$ |
| $\rho$ | $\frac{1}{1-\rho}$ | $\frac{\rho-1}{\rho}$ | $1-\rho$ | $\frac{1}{\rho}$ | $\frac{\rho}{\rho-1}$ |
| $1$ | $l_1 l_2$ | $l_2 l_1$ | $l_1$ | $l_2$ | $l_1 l_2 l_1$ |
| $1$ | $(a_1 a_3 a_2)$ | $(a_1 a_2 a_3)$ | $(a_2 a_3)$ | $(a_1 a_3)$ | $(a_1 a_2)$ |

The fourth row of this array represents the group formed by the six anharmonic ratios of four points, while the fifth and sixth rows represent the group of movements of an equilateral triangle, and the symmetric group of degree three respectively, $l_1$ and $l_2$ representing rotations through $\pi$ any two lines of symmetry of the triangle. The last two rows are added mainly to emphasise the known fact that the same group properties present themselves in such different subjects, each of these rows being simply isomorphic with the first. The first and fourth rows seem to exhibit most clearly how the squares of the hyperbolic functions, and hence these functions themselves, may be obtained from any one of them. In this array they are derived from $\operatorname{sech}^2 x$, but they could just as readily have been derived from any other one of the set.

The second and third rows, in connection with the first, exhibit a method of expressing all these hyperbolic functions in terms of an arbitrary one of them in exactly the same manner as the trigonometric functions are

expressed in terms of one of them. If the functions in these rows are supposed to imply the operation by means of which each of them has been derived from the first one in the row, these lines may be regarded as representing the symmetric group simply isomorphic with that represented by each of the other rows. From the table above it is evident that *the values of the squares of the six hyperbolic functions of $x$, as well as of the six trigonometric functions of $x$, are for any value of $x$ conjugate numbers with respect to the group of subtracting from unity and dividing unity provided that a properly chosen pair of them is taken with the negative sign.*

It is of some interest to observe that $\coth^2 x$ and $-\sinh^2 x$ may be derived from $\operatorname{sech}^2 x$ by an operation of period three under $\{s_1, s_2\}$, while each of the remaining three functions in this row may be derived from $\operatorname{sech}^2 x$ by an operation of period two under this group. That is, the operation under $\{s_1, s_2\}$ which transforms $\operatorname{sech}^2 x$ into $\tanh^2 x$, for instance, must also transform $\tanh^2 x$ into $\operatorname{sech}^2 x$. Similar remarks apply to $\cosh^2 x$ and to $-\operatorname{csch}^2 x$ but not to $\coth^2 x$ or to $-\sinh^2 x$. With respect to the cyclic subgroup of order three $(1, s_1 s_2, s_2 s_1)$, the first three functions of the second row form a complete set of conjugates. Thus we see that *the six squares of the hyperbolic functions as given in the table are divided into two sets of three each. Any one of either set may be transformed into one of the same set by means of an operation in the cyclic subgroup* $(1, s_1 s_2, s_2 s_1)$. *Any one of the first set may be transformed into any one of the second set by* $s_1$, $s_2$ *or* $s_1 s_2 s_1$. Each of the last three operators is of period two and interchanges these two sets. From what precedes, it is clear that the hyperbolic functions, the trigonometric functions, the anharmonic ratios, and any other similarly related numbers may be obtained from each other by the same fundamental operations.

Since the hyperbolic functions have the period $2\pi i$, the two operations $\frac{\pi i}{2} - \alpha$, $\pi i - \alpha$ may be associated with those of taking the complement and the supplement of an angle. The octic group generated by these two operations may therefore be employed in the study of the hyperbolic functions in practically the same way as it was employed in the study of the trigonometric functions in an Article published in this Journal, volume III, No. 59. The present note may be regarded as a continuation of this Article. Similar remarks hold as regards the other groups of subtraction which have been used to find values of trigonometric functions. Cf. *Annals of Mathematics*, volume 8 (1907), p. 97.

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## DECIMALS.

MR. GRANT, in the last issue of this *Gazette*, having made several references to "Arithmetical Types and Examples," perhaps a few remarks in reply will not be altogether out of place.

1. *Multiplication.* The only difference between the two methods first given is in the position of the various digits of the multiplicand.

We are told that the first rule (that "there are the same number of figures to the right of the decimal point in the first partial product as there are in the multiplicand," and that therefore the first figure set down must be under the right-hand figure of the multiplicand) is *mechanical*; whereas the rule of "putting each first figure of a partial product underneath the figure to which that partial product is due" is *sound* and *fundamental*. It is not obvious to the writer that one is more mechanical than the other.

The explanation immediately afterwards given by Mr. Grant of course applies to both methods of working. The reason for the position of the first figure of the first partial product in the first method is contained in the note to Ex. 1, p. 69, in the statement that "we are multiplying by 8 *units*,"

and that therefore five thousandths multiplied by 8 gives 40 thousandths or 4 hundredths and *0 thousandths*, and thus 0 is put in the thousandths column, and consequently *under the right-hand figure of the multiplicand*; it is intended that this reasoning, or something similar, should be supplied by the teacher. The volume referred to does not profess to be a complete text-book, but "explanatory notes are added where necessary; the proof of the method, in many cases, is purposely left to be supplied by the teacher."

We are told that the second method "corresponds to the method of algebraic multiplication." Does Mr. Grant mean that the product of $x+4$ and $x+5$ is usually put down

$$\begin{array}{r} x+4 \\ x+5 \\ \hline x^2+4x \\ 5x+20 \\ \hline x^2+9x+20 \end{array}$$

*i.e.* that $4x$ in the first partial product is usually placed under $x$ of the multiplier? The writer cannot find that text-books support this view; but perhaps he does not understand Mr. Grant's point.

The method of reducing the multiplier to "standard form" is—as pointed out in the article under discussion—advocated in the Report on "The Teaching of Mathematics in Preparatory Schools," but not knowing who constituted the Committee, the writer cannot say what their reasons were for this recommendation. It should however be noticed that

(i) If the multiplier or divisor is written in "standard form," then a rough answer can very readily be obtained by taking this multiplier or divisor to the nearest unit, and consequently determining the position of the decimal point in the product or quotient by inspection.

(ii) It is useful to be able to write large or small numbers in this way, particularly when dealing with physical constants.

(iii) It is useful in the method suggested for Contracted Multiplication and Division in "Arithmetical Types and Examples," since it gives a very simple method for determining at the outset the number of figures of the multiplicand it is necessary to retain.

The writer is an advocate for the retention of Contracted Work, since he has found that boys are very soon able to use the method with facility, and as a matter of personal experience, he frequently uses it incidentally when it is not worth while using Logarithm Tables, and when, moreover, he requires a greater degree of accuracy than 4-figure tables permit.

Why object to the phrase "moving figures to the right"? It would naturally be explained by the teacher that this means that the number has been divided by a power of 10 and that this is effected by altering the places of the digits relatively to the decimal point.

2. *Division.* Mr. Grant says " . . . the plan is abandoned and the divisor is made such a number as to contain one single's figure." What plan? Arranging the divisor in standard form? But apparently he does not mean this, for in the same sentence Mr. Grant admits that the divisor *is* arranged in standard form. The method of standard form is, as a matter of fact, abandoned in one particular type of examples, and another method is suggested as being "more convenient," when the significant figures of the divisor can be split up into a small number of *simple factors*, as in the example given:

$$\frac{\cdot00406}{\cdot056}=\frac{4\cdot06}{56}=\frac{\cdot58}{8}=\cdot0725.$$

But in the case which Mr. Grant seems to think analogous, viz. $12\cdot45\div\cdot83$, the divisor has no factors, and the writer would certainly adopt the standard

form method of division. Mr. Grant moreover admits that "the factor method" is advisable sometimes, since he suggests that

$$\frac{40{\cdot}96}{{\cdot}032}=\frac{40960}{32}=\frac{5120}{4}=1280.$$

Lastly, in Contracted Work, the writer thinks that the *whole point* is *to devise a simple mechanical method*, since, for all practical purposes, it is only used incidentally in the solution of other problems where it is desired to make such calculations as quickly as possible and with mechanical accuracy.

W. G. BORCHARDT.

## ON SOME EARLY PROPOSITIONS OF STATICS.

THE current method of founding the teaching of Theoretical Statics on the experimental proof of the "Parallelogram of Forces" seems open to grave objection. The experiment is too remote from the facts of ordinary observation; and it is not so very easily performed. Moreover, the result, when accurately obtained, appears to the pupil to be so remarkable, that it is a pity not to reserve it for use as a verification of the soundness of pure reasoning.

The alternative treatment of the elementary bookwork which is given below is based, experimentally, on a simple fact of universal knowledge and daily experience. It possesses two additional advantages; *i.e.* it leads at once to perhaps the easiest type of numerical examples, viz. those on parallel forces and moments; and it gives the pupil the sense of 'power' by arriving very early at the "Principle of the Lever."

It is suggested that the following seven propositions should be taken in the order named.

I. *Experimental basis.* At the same place on the earth's surface equal volumes of the same material have equal weights.

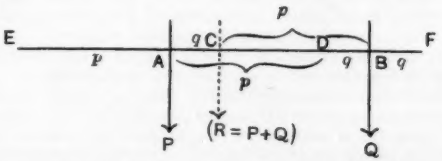
II. The weights of bodies of the same material are proportional to their volumes.

[Follows from I., by addition.]

III. The weight of a uniform rigid *horizontal* rod may be supposed to act through its middle point.

[*Proof.* If supported solely at its middle point, it will remain at rest, ∵ there is no reason why it should drop on one side more than on the other. Hence its resultant weight is equal and opposite to the pressure of the support.]

IV. To find the resultant $R$ of two like parallel forces $P$ and $Q$, acting on a rigid body.




Let any straight line $AB$, drawn $\perp^r$ to the lines of action of $P$ and $Q$ and terminated by them, be divided at $C$ and $D$, so that

$$BC : CA :: AD : DB :: P : Q.$$

Let $AD=BC=p$ units of length, $BD=AC=q$ units of length.

(1) Suppose the system held so that the directions of $P$ and $Q$ are vertically downwards.

(2) Suppose $P$ and $Q$ to be due respectively to the weights of two uniform horizontal rods $EAD$, $DBF$ (lengths $2p$ and $2q$ respectively) *rigidly fixed* to the body (II. and III.).

(3) We now have a single uniform rigid horizontal rod $EF$, length $2p+2q$, with its centre at $C$ ($\because EA+AC=p+q$).

The weight of this rod $=P+Q$ (II.), and acts vertically through $C$ (III.).

$\therefore R=P+Q$ and acts in the same direction as $P$ and $Q$, through a point $C$, such that $P.AC=Q.BC$.*

V. The "moment" (as ordinarily defined and represented) of a force about a point is a true measure of its turning effect about that point.

[*Proof.* Let $P$, $Q$ have equal moments $Pq$, $Qp$ in the same sense round $O$.

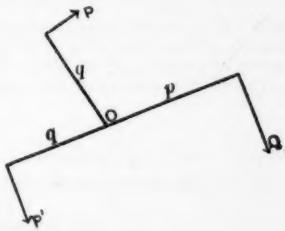
Consider a force $P'$ ($=P$) acting in the same direction as $Q$, but with moment $P'q$ equal and opposite to that of $Q$ round $O$.

The turning effect of $P'$ round $O$ would balance that of $Q$, $\because$ their resultant passes through $O$ (IV.).

But, by symmetry, the turning effect of $P'$ would balance that of $P$.

$\therefore$ the turning effect of $P$ round $O$ = the turning effect of $Q$.

$\therefore$ the moment is a true measure of the turning effect.]

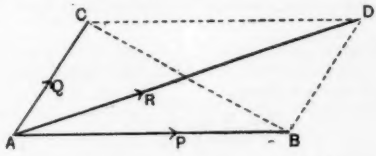



VI. The moment of the resultant of any number of coplanar forces acting on a rigid body about any point is equal to the algebraical sum of the moments of the components about the same point.

[*Proof.* By definition, the resultant has in every way the same effect as that of the forces combined.

$\therefore$ this proposition is an immediate corollary of V.]

VII. To prove "The Parallelogram of Forces."




Let two forces $P$, $Q$, acting at a point $A$, and their resultant $R$, be represented in magnitude and direction by the straight lines $AB$, $AC$, $AD$ respectively.

Then the moment of $R$ about $B$ = the sum of the moments of $P$ and $Q$ about $B$ (VI.).

$$\therefore 2\triangle ABD = 0 + 2\triangle ABC;$$

$$\therefore \triangle ABD = \triangle ABC;$$

$\therefore CD$ is parallel to $AB$.

Similarly, $BD$ is parallel to $AC$.

$\therefore ABDC$ is a parallelogram. Q.E.D.

W. E. BRYAN.

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* This proof was in substance communicated to me by my father, the Rev. R. G. Bryan, M.A., who invented it in 1841.

## MATHEMATICAL NOTES.

**285.** [S. 1. b.] *Stability of cube floating in liquid.*

It is sometimes assumed in elementary questions on flotation that a cube necessarily floats with a face uppermost. There are, however, two other positions of symmetry about a vertical axis, when an edge or a vertex is uppermost; and these must be positions of equilibrium.

The equilibrium is stable or unstable according as $GM$ is positive or negative; $G$ being the centre of gravity and $M$ the metacentre.

$$GM = B + \frac{Ak^2}{V} - C,$$

where $B$, $C$ are the heights of the centres of buoyancy and gravity above the lowest point of the cube, $V$ is the volume immersed, and $A$, $k$ the area and radius of gyration of the water-line section.

The specific gravity of the cube relative to the liquid is given by

$$s = \frac{V}{a^3},$$

where $a$ is the edge of the cube.

(1) If a face is uppermost, these formulae become

$$GM = \frac{1}{12b}(a^2 - 6ab + 6b^2)$$

and

$$s = \frac{b}{a},$$

where $b$ is the height of the water-line above the lowest point of the cube.

Hence

$$\frac{GM}{a} = \frac{1}{12s}(6s^2 - 6s + 1),$$

shewing that this position is very stable for small values of $s$, becomes unstable when $s$ passes 0·211, becomes stable again at $s = 0{\cdot}789$, and remains so until $s = 1$.

(2) If an edge is uppermost, there is a geometrical discontinuity at $s = 0{\cdot}5$; there are statical discontinuities at $s = 0{\cdot}25$ and 0·75, for at these values the axis about which $k^2$ is least changes.

From $s = 0$ to 0·25, $\dfrac{GM}{a} = \dfrac{4}{3}s^{\frac{1}{2}} - \dfrac{1}{\sqrt{2}}$;

$s = 0{\cdot}25$ to 0·5, $\dfrac{GM}{a} = \dfrac{2}{3}s^{\frac{1}{2}} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{6}s^{-\frac{1}{2}}$;

$s = 0{\cdot}5$ to 0·75, $\dfrac{GM}{a} = \dfrac{1-s}{s}\left\{\dfrac{2}{3}(1-s)^{\frac{1}{2}} - \dfrac{1}{\sqrt{2}} + \dfrac{1}{6}(1-s)^{-\frac{1}{2}}\right\}$;

$s = 0{\cdot}75$ to 1, $\dfrac{GM}{a} = \dfrac{1-s}{s}\left\{\dfrac{4}{3}(1-s)^{\frac{1}{2}} - \dfrac{1}{\sqrt{2}}\right\}$.

It follows that this position is never stable, though it is less unstable than the first position between $s = 0{\cdot}249$ and $s = 0{\cdot}751$, and becomes neutral at $s = 0{\cdot}5$.

(3) If a vertex is uppermost, there are geometrical discontinuities at $s = \frac{1}{6}$ and $s = \frac{5}{6}$; the water-line section, which is an equilateral triangle in the first and last regions, being truncated at its vertices in the middle region so as to become a trisymmetrical hexagon. Throughout the three regions $k^2$ has six equal minimum values, so that there are no statical discontinuities.

From $s=0$ to $\frac{1}{6}$, $\frac{GM}{a}=\frac{\sqrt{3}}{2}\{(6s)^{\frac{1}{3}}-1\}$;

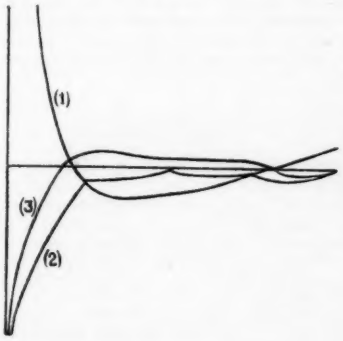
$s=\frac{1}{6}$ to $\frac{5}{6}$, $\frac{GM}{a}=\frac{p}{4\sqrt{3}s}(p^2-6p+6s)$;

and $s=\frac{1}{6}(p+1)^3-\frac{1}{2}p^3$, where $p=\sqrt{3}\frac{b}{a}-1$;

$s=\frac{5}{6}$ to 1, $\frac{GM}{a}=\frac{\sqrt{3}}{2}\cdot\frac{1-s}{s}\{6s-6)^{\frac{1}{3}}-1\}$.

This position is therefore stable between $s=0{\cdot}167$ and $s=0{\cdot}834$, and exceeds the first position in stability between $s=0{\cdot}191$ and $s=0{\cdot}808$.

In the accompanying figure, $\frac{GM}{a}$ is plotted vertically against $s$ horizontally.




To sum up, the positions of stability are:

$s=0$ to $0{\cdot}191$, $s=0{\cdot}808$ to 1, face uppermost.

$s=0{\cdot}191$ to $0{\cdot}211$, $s=0{\cdot}789$ to $0{\cdot}808$, face or vertex uppermost.

$s=0{\cdot}211$ to $0{\cdot}789$, vertex uppermost.

$s=0{\cdot}5$, position of edge uppermost is neutral.

Surface tension has here been neglected. In practice the above results will be found to hold good, with possible exceptions in the neighbourhood of the intersections of the curves if the cubes are small. C. W. ADAMS.

**286.** [V. 1. a. v. Note 254, p. 231.]

We certainly need a sign to denote approximate equality. Probably the best sign will be determined by natural selection. The one suggested by C. S. J. is easy to write, but does not look neat. The only other one I have seen is $\backsim$, which is used by O. Lodge (*Easy Mathematics*); the sign is convenient, in that a "=" written by mistake is easily altered, but the reason for choosing the sign is not clear. How would it do to combine the signs $>$ and $<$, rounding the angles? One portion might be made greater than the other, if we knew which of the quantities was the greater; thus "$a \rtimes b$" would mean "$a$ is approximately equal to $b$," and "$a \ltimes b$" would mean "$a$ is approximately equal to $b$, but less than it." W. F. SHEPPARD.

**287.** [D. c. v. Note 281, p. 332.] The excellent note on Taylor's Theorem by Professor Harding makes the whole investigation wonderfully clear and simple. It might be slightly improved by writing $a$ instead of $x$, throughout, so making it clear that $z$ is the only variable, used for a sort of search-light on the variation of the function in the interval between $x=a$ and $x=a+h$. The words "now form a new function of $2^n$ might read:—now let $z$ be a quantity whose values range from 0 to $h$, and form a function of $z$ from the right hand side of the above, by writing $a+z$ for $a$, and $h-z$ for $h$: etc. etc.

A. LODGE.

**288.** [R. 6. v. Note 275, p. 327.] The pretty note on loss of energy by collision would, I think, be improved by calling $V_1$ "the velocity of approach," and letting $V_2$ be "the velocity of separation," so that $V_2$ would be the essentially *positive* quantity $V_2-u_2$ instead of the essentially *negative* one, viz. $u_2-V_2$, in the text. Thus $V_1$ and $V_2$ would both be positive. This would require us to write $V_1+V_2$ where now $V_1-V_2$ occurs, and $V_1-V_2$ where $V_1+V_2$ occurs, and would make $V_2=eV_1$ instead of $=-eV_1$. It is perhaps a small matter, but I think it is better for the symbols to denote positive quantities where possible, so that a minus sign indicates a real subtraction instead of a veiled addition, which obscures the real inwardness of the formula.

A. LODGE.

## REVIEWS.

**Fourier's Series and Integrals.** By H. S. CARSLAW, D.Sc. Pp. xvii+434. (London: Macmillan & Co., 1906.)

The simplest attempt at a proof of Fourier's Theorem arises when it is sought, in the first place, to construct a trigonometrical series which shall have given values at $n$ points uniformly distributed round a circle. If the angular co-ordinates of these points are $0, A, 2A, 3A$, up to $(n-1)A$, where $nA=360°$, we commence by finding a series which is equal to unity at the point $\theta=0$, and to zero at all of the other points.

Now, if we take $n$ unit vectors whose angular coordinates are $0, \theta, 2\theta, 3\theta$, up to $(n-1)\theta$, and lay them end to end, they will form a closed figure when $\theta$ has any one of the above values except zero. In the latter case they will lie in a straight line of length $n$. Projecting on the initial line and at the same time dividing by $n$, we see that

$$\frac{1+\cos\theta+\cos 2\theta+\ldots+\cos(n-1)\theta}{n}$$

is equal to 1 when $\theta=0$, and equal to zero at the other points.

Substituting $\theta-A$ for $\theta$, we get an expression which is equal to 1 when $\theta=A$, and equal to 0 at the other points, this expression being

$$\frac{1+\cos(\theta-A)+\cos 2(\theta-A)+\ldots+\cos(n-1)(\theta-A)}{n}.$$

If we now try to build up an expression which equals $F(0)$ at $\theta=0$, $F(A)$ at $\theta=A$ and so on, we must multiply the first of the above by $F(0)$, the second by $F(A)$ and so on, and add. When we expand the compound angles such as $\cos(\theta-A)$, $\cos 2(\theta-A)$, $\cos(\theta-2A)$ and so on, we get a series of sines and cosines of multiples of $\theta$, in which

$$\text{the constant term}=\frac{F(0)+F(A)+F(2A)+\ldots}{n}$$

$$=\text{mean value of } F(\theta) \text{ at the several points;}$$

the coefficient of $\cos r\theta$ is

$$=\frac{F(0)+F(A)\cos rA+F(2A)\cos 2rA+\ldots}{n}$$

$$=\text{mean value of } F(\theta)\cos r\theta \text{ at the several points;}$$

and similarly, the coefficient of $\sin r\theta$ is equal to the mean value of $F(\theta)\sin r\theta$ at the several points.

Unfortunately, however, this is not Fourier's Series, and it is a great pity that we cannot insert or leave out a factor $\frac{1}{2}$ where we want it, to get the right answer. For, calling the final result

$$F(\theta) = \Sigma A_r \cos r\theta + B_r \sin r\theta,$$

if we multiply both sides by $\cos r\theta$ and take mean values for the $n$ points round the circle, the mean value of the squared cosine is equal to $\frac{1}{2}$, and gives

$$\tfrac{1}{2}A_r = \text{mean value of } F(\theta)\cos r\theta$$

or $A_r$ = double the value previously obtained.

We should like to see the mistake in the reasoning pointed out in print in the columns of the *Gazette*. Meanwhile, the illustration may serve to prove how easy it is to make mistakes in dealing with such a subject as Fourier's Theorem, and how necessary it is to keep in close touch with the actual literature of the subject.

Only two proofs of Fourier's Theorem have hitherto found their way into English text-books. Neither of these is usually dealt with in such a way as to satisfy modern demands for rigour. These demands have, in fact, made it necessary for Prof. Carslaw to write what may be described as an *ab initio* treatise. Such a treatise can be not unsuitably described as an introduction to higher analysis, and accordingly we find the first four chapters devoted respectively to rational and irrational numbers, infinite sequences and infinite series, the theory of infinite series and the convergence of infinite definite integrals.

We should have liked to see more attempt at illustration by geometrical examples, notably in discussing such matters as principal values and differentiation of integrals. It is not difficult to give some elementary explanation of principal values in connection with the integration of rational fractions. It is not difficult to give even to beginners in the calculus some idea of the significance of the principal value of such an integral as that of $(x^2-1)^{-1}$ when carried past the value $x=1$, especially if the areas of the graphs of the two partial fractions are treated separately. There are a great many students who are made to spend much time in evaluating far more complex integrals, and for want of this knowledge their results cannot be described as much better than mere uncomprehended cram. The remedy is to spend less time on the mere mechanical drudgery of integration, and if possible to devote some of the time to discussion of the meaning of the results. If Dr. Carslaw should say that such matters are dealt with adequately in other books, we could not agree, but he undoubtedly has an unassailable position if it is claimed that this introductory matter only leads up to the real subject of the book.

The next four chapters deal with Fourier's series and integrals. They may certainly be stated to contain the best guide to the subject that has been published in England.

The second part consists of applications to the solution of the equation of conduction of heat. In it the author has discussed problems such as form the subject of Fourier's classical *Théorie de la Chaleur* by the rigorous methods of modern analysis. The historic introduction at the beginning of the book may be studied with great interest in this connection. It shows how Fourier discovered results of his own, and further made use of trigonometrical series previously employed by Bernoulli, Lagrange and Euler, but the rigorous proofs were not forthcoming till a much later date. Prof. Carslaw's treatment embraces the usual problems of flow of heat in one, two and three dimensions with plane boundaries, cylinders and spheres, the method of sources and sinks and applications of Green's Functions.

The book is eminently suited to the mathematical student. As regards the student of physics, we would counsel him to have a copy of it in his possession. It is not always, if ever, possible for the physicist to go through the whole of the mathematical drill required to prove the methods of reasoning about such problems as occur in the conduction not only of heat but also of electricity. It would be equally impossible for the mathematician to perform the actual experiments required to prove the laws of dynamics and hydrostatics which he uses to exemplify his work in pure mathematics. What is wanted in either case is the possession of a book in which the details of this preliminary work are down in black and white so that the student may satisfy himself that the work has been

done, and if he wishes, untangle the train of reasoning by reading backwards as each result is made to depend on something previous.

Notable features are the collections of examples. Some of these look fairly simple, though others have rather a Cambridgy twang. Also the bibliographies both at the end of the chapters and at the end of the book. These are very exhaustive. It is impossible for a reader to form an estimate of the extent to which the contents of the book represent original work on the part of the author. This could only be done by reading carefully through all the memoirs cited. But we have a shrewd idea that Dr. Carslaw is responsible for a great many of the methods adopted.

G. H. BRYAN.

**Algebra for Secondary Schools.** By C. DAVISON, Sc.D. 623 pp. (Cambridge University Press.)

**Elementary Algebra.** By C. H. FRENCH, M.A., and G. OSBORN, M.A. 506 pp. (Cambridge University Press.)

**Elementary Algebra for the use of Higher Grade and Secondary Schools.** By P. ROSS, B.A., B.Sc. 548 pp. (Longmans, Green & Co.)

**School Algebra,** Part I. By W. E. PATERSON, M.A., B.Sc. 367 pp. (Oxford : Clarendon Press.)

**The Eton Algebra,** Part I. By P. SCOONES, M.A., and L. TODD, M.A. 184 pp. (Macmillans.)

**Blackie's Elementary Modern Algebra,** in three parts. By R. C. BRIDGETT, M.A., B.Sc., F.C.S. 231 pp. (Blackie & Son.)

**A New Algebra,** Vol. I. By S. BARNARD, M.A., and J. M. CHILD, B.A., B.Sc. 371 pp. (Macmillans.)

Of making many Algebras there is no end; and much reading of them is a weariness of the flesh. It is rather difficult to account for the simultaneous advent of all these books, in which only one has any very special features differing from those of the text-books already in general use. Most of them recognise the difficulty of bringing the book-work down to the level of beginners, and an attempt is sometimes made to meet this by making the language of a simple and even colloquial type. We do not think this is a step in the right direction, for we take it that no teacher would dream of putting any text-book on algebra into the hands of a pupil who is innocent of the subject, and expect him to wrestle with it, and to ask for an explanation of any difficulties that arose. The only practical way, and we think the one almost universally adopted, is for beginners to approach the subject chiefly through the medium of the teacher and the blackboard, and for the teacher to take them along the lines of the particular text-book which he is using.

Another point in which we think a mistake is made is the enormous number of examples that are given. Of course a certain amount, we might even say, a large amount, of practice is necessary to impress the principles and rules upon the young pupil, but this is very liable to be overdone, and the hundreds, we might even without exaggeration say thousands, of examples through which he is expected to plough prevents his interest from ever being roused on the subject. Of course we do not advocate a return to the type of text-book which was all theory without any examples, but we do think that with fewer exercises the student would see more of the subject, and the variety would have a more stimulating effect upon him. It must also be remembered that algebra is only one of a very large number of subjects that occupy the young student, and he simply has not the time to give to it which many of our text-book writers seem to expect.

In Dr. Davison's *Algebra for Secondary Schools*, in addition to some thousands of examples we find at intervals through the book a large number of sets of exercises intended to be answered *viva voce*, which strikes us as a very useful feature, and a good deal of pains is taken in teaching students to translate the words of a problem into algebraical language, in other words, to think for himself. Graphs, now a necessary feature in every elementary algebra, are introduced and employed in the solution of simultaneous equations, and also in indeterminate equations, which enables the latter to be considered at a much earlier stage

than usual. It is a pity that any stress should be laid upon the solution of a quadratic equation by substitution in the formula, which is often the source of inaccurate results, and is rightly penalised in many examinations. There is a very good chapter on the graphs and theory of quadratic equations and expressions, and the Binomial Theorem is thoroughly treated. There is also a short chapter on exponential and logarithmic series, followed by another on miscellaneous graphs. The student who could find the time to work through the book would certainly be well equipped for higher work. We would call the author's attention to the repetition of the equations in Ex. 17, p. 221, and in Ex. 11, p. 495, an important part of the statement of the question is omitted.

In the *Elementary Algebra*, by Messrs. French and Osborn, the writers have arrived at simplicity of style, especially in the earlier chapters, and they have certainly taken great pains to make the path as smooth as possible for beginners, and a good many examples are given for oral use. The Remainder Theorem is introduced at the end of division, so as to be ready for use in factorising expressions of higher degree than the second. Graphs are used to solve simple simultaneous equations, and also to determine statistics of different kinds. The method of detached coefficients is explained in the chapters on Multiplication and Division, but although the authors point out its value they have not the courage of their opinions to employ it in the examples on H.C.F. where so much wearisome writing of terms with their indices distract the attention of the worker from the real point at issue. It is surprising how little work there is in even the longest H.C.F. when only the coefficients are employed, especially when the student is taught to write down only the remainders, which of course are all that are of importance. Quadratics are well treated, both theoretically and graphically. There is a good introduction to common logarithms, with examples on their applications, and a chapter on the Binomial Theorem for a positive integral index. The examples throughout the book are very numerous, and many of them are of an interesting nature.

The two previously mentioned works are on what we may call the old-fashioned lines of which Todhunter's larger edition was the type. Mr. Ross, in his *Elementary Algebra*, has elected to follow the lead set by Prof. Chrystal ten years ago in his Introduction to Algebra. Without giving too much book-work he introduces the beginner at a very early stage to fundamental ideas which are of practical interest. Graphical work is introduced almost at the commencement, and is used to solve equations, find areas, calculate statistics, and answer the various practical questions which present themselves in laboratory work. In the advanced part of the book the graphs of rational functions have been discussed by change of origin and approximations, and by this thorough treatment the student obtains a clear notion of a continuously varying function, a limit, and an approximation to it. Indeterminate coefficients are introduced and freely employed in factorisation, expansions, simplifications and partial fractions. There is also a useful chapter on Limits and Infinite series, and it contains such theorems on convergency as are needed in discussing the Binomial, Exponential, and Logarithmic Series.

Mr. Paterson's *School Algebra*, Part I., covers the ground as far as simultaneous quadratics and a few equations of higher degree, together with a chapter on logarithms. The ground-work of the subject is very carefully laid, the explanations of the different steps and processes being given in very simple language, but we do not quite see the author's reason for giving a chapter on problems before he has taught the student how to solve the equations on which they depend. In the chapter on Division we meet with Horner's method, and considerable use is made of the Remainder Theorem in factorisation. Graphs are introduced, but not very freely, being chiefly employed in the solution of equations above the first degree. The book concludes with a large number of miscellaneous questions and examination papers.

*The Eton Algebra*, by Messrs. Scoones and Todd, consists of about 2600 examples of the usual type. The preface tells us that it is "for the use of those commencing the study of Algebra at Eton. No book work has been considered necessary in these early stages, all explanation being left to the teacher." The questions include graphs and quadratic equations, and in the event of a teacher

finding that the questions in the text-book which he was using gave out, he would be able to supplement them from the Eton Algebra.

Mr. Bridgett's little book, *Elementary Modern Algebra*, would be best described as a collection of elementary examples up to and including quadratics, with occasional rules for working them. Graphical work constitutes a large part of the book, and is applied to the solution of various kinds of arithmetical and geometrical problems, simultaneous and quadratic equations, square roots and reduction of fractions. The examples are numerous and suited for beginners.

In *A New Algebra*, Vol. I., by Messrs. Barnard and Child, we have a distinct effort to get out of the beaten track in the logical development of the subject. The present volume consists of 3 parts, in the first of which the letters employed denote natural numbers, and the principles underlying the four simple rules are carefully explained subject to this limitation. In Part ii. we are introduced to zero and negative numbers, and their meanings are explained in accordance with modern views, and it is shown that although they have no cardinal property, they have their place in the natural scale, and the laws of addition, etc., are extended so as to include them. Whilst the authors were dealing with zero as a multiplier they might have given an explanation of the meaning of $0 \times 0$ on p. 175, about which an enquiring student might ask awkward questions. In Part iii. it is shown that fractions are rational numbers which also belong to the same scale, and the meaning of each class of number with respect to measurement is separately considered. In establishing the various fundamental operations stress is laid upon the Permanence of Algebraic Form. Graphs are introduced in connection with simple simultaneous equations, and are applied to the solution of various classes of questions. The present volume deals with quadratic equations and problems connected with them, and concludes with a number of test papers in which many of the questions are of a distinctly stimulating character. We have no hesitation in saying that the book is one which will make its mark in the world of school, and we should be glad of the opportunity of using it with an intelligent class. We shall look forward with interest to the appearance of the second volume, in which the authors propose to carry the subject up to the Binomial Theorem. JOHN J. MILNE.

**Practical Mathematics.** By F. C. CLARKE. Pp. with Answers and Tables, 270. 3s. 6d. (Ed. Arnold, 1907.)

There is a tendency nowadays on the part of certain people to talk as though the mathematics required by the "practical man" is essentially different from that required by the academic mathematical specialist, and they often jeer at the latter as a dealer in a rather useless kind of knowledge. No doubt such extremists are partly right and may have sounded a useful note of warning, but the perusal of a few books on what is called "practical mathematics" shows that both the academic student and the practical man must journey together for some way on the road to mathematical learning. Thus in the book before me there are 50 pages each on elementary Algebra and Trigonometry, 30 on Mensuration, 50 on Calculus, 40 on Graphs, and some odds and ends. Except for pp. 125-180, on Graphs and Mensuration of Irregular Figures, there is very little in the book which would not have been done by the most academic student in those bad old days which are now supposed to be ended, and the corresponding modern schoolboy would know nearly all these 55 pages referred to; even among the examples there are scarcely any which do not appear in old-fashioned text-books.

The fact is, that the "practical" man does not need anything different from the academic student; he needs less, but wants a little extra practice with squared paper. It is merely the words "practical mathematics" that are misleading, because we often think of theory and practice as opposites. But the appearance of such books as this merely confirms the view that practice and theory cannot be divorced,—that, in fact "Practical Mathematics" is a misnomer—it ought to be "Theoretical Mathematics which a Practical Man cannot do without."

To attend to this particular book after a preliminary grumble—the most noticeable thing is that each subject is cut down to its minimum, and as a consequence there is naturally nothing original or striking in the methods employed. The best part of the book is that on the use of squared paper referred to above.

The Calculus is confined to 50 pages by assuming all derivatives except that of $ax^n$. Speaking from experience, I should say that the idea of Integration as a Sum requires much more explanation than is given, and for weak students it is better to omit it.

In the short chapter on vectors no indication of the utility of vector and scalar products is given. W. M. R.

**An Introduction to Practical Mathematics.** By F. M. Saxelby. Pp. 220, including answers and tables. Price 2s. 6d. (Longmans, Green & Co.)

This book is written as an introduction to mathematics for students who know little more than some arithmetic and mensuration. It contains, therefore, the beginnings of both algebra and trigonometry and a good deal of work on squared paper. The examples are as far as possible chosen from engineering or physics. The author has not attempted to compress his subjects into too small a compass, and important points receive a proper share of attention, *e.g.* the question of degrees of accuracy in arithmetical operations in Chap. I., the early introduction of approximation in algebraical multiplication by using $(1+x)^n=1+nx$ when $x$ is small (pp. 51-4); the precautions as to scale in plotting, p. 92. The presentation of the more commonplace parts of the subject is interesting. The book is generally well written, and appears to be admirably adapted to the needs of the students for whom it is designed. We recommend a perusal of the preface as giving the author's 'confession of faith.'

W. M. R.

**Graphic Algebra.** By Arthur Schultze, Ph.D. Pp. 72+appendix, tables and answers. (New York: The Macmillan Co.)

This little book deals chiefly with the graphical solution of equations up to the 4th degree. The last 50 pages or so are concerned with solutions by means of $y=x^2$ and $y=x^3$ as standard curves. Cubics are solved by finding the intersection of the latter with a straight line and biquadratics by intersection of the former with a circle. Alternative methods are given and also a number of ingenious methods, including the finding of imaginary roots. Incidentally the pupil is taught a little analytical geometry. It is a little difficult to see what class of pupil the book is intended for, as it gives so many special methods, which will give a lot of trouble to the average boy to understand. W. M. R.

**Computation and Mensuration.** By P. A. Lambert, M.A., Professor of Mathematics in Lehigh University. Pp. ix, 92. Price 3s. 6d. net. (Macmillan & Co., 1907.)

This little treatise would make a capital supplement to some of the older books on algebra and mensuration. This is indicated by the subjects of the ten chapters—viz.:

Approximate computation, graphic computation, the method of coordinates, volumes of solids bounded by planes, trigonometric functions, logarithms, limits, graphic algebra, areas bounded by curves, volumes of solids.

The principles of contracted methods are made plain, but the objectionable plan of reversing the order of the figures in the multiplier is adhered to. Professor Alfred Lodge's arrangement (*Gazette*, iv. 17) is decidedly preferable.

The Simpson-Weddle types of formulae are explained, and the reason why these formulae give *exact* results in certain cases is clearly shown.

We can only hope that American authors benefit by the price of American books. C. S. J.

**Leçons élémentaires sur le calcul des probabilités.** Par R. de Montessus. Pp. 191. Price, 7 fr. 1908. (Paris: Gauthier-Villars.)

The association of the subject of Probability with the disreputable names of chance and the odds, and a natural desire to diminish the randomness of the collection which goes by the name of higher algebra, have deprived the theory of a place in ordinary programmes of instruction.

This is in some respects to be regretted, for no branch of mathematics is increasing more rapidly in practical importance in connection with the treatment of statistics; and none demands greater precision in the use of the English language. The student of probability learns to distinguish between the average

height of a number of men, the height of the man who occupies the middle place when all are drawn up in order of height, and the most fashionable height, the height which is common to the greatest number of the men; but each of these may be loosely spoken of as the mean height.

The moralist who objects to association with the tools of the gamester may be asked to consider whether or not a 'sporting' schoolboy would have his eyes opened by a demonstration from the odds in the morning's paper that the backer of horses is playing against loaded dice, and is, if he bets for any reason save to get rid of his superfluous cash, merely a fool, pitiable because of his ignorance.

The advocates of 'practical mathematics' have incurred censure, not perhaps altogether undeserved, for depriving school mathematics of its backbone and reducing it to a disconnected series of tricks. But the real aim of the mathematical radicals is to combine logical order with concrete illustration.

It is submitted that as regards probability the true reconciliation of conflicting views is not to be found in dropping the subject; nor in giving an hour or two to it as a detached item out of all relation to past or future work; but by introducing it when opportunity offers, as a simple practical application of pure mathematics.

For instance, when the binomial theorem is reached we may symbolise the possible results of tossing a penny thus: $H+T$, of two tosses $HH+HT+TH+TT$, or, for some purposes,

$$H^2+2HT+T^2 \text{ or } (H+T)^2,$$

and the binomial theorem for a positive integral index becomes a table of the possibilities of $n$ throws of a coin.

The distinction which so fatally eludes the gambler between the chance of a named sequence of heads and tails, say $HHT$, and of a specified total number of heads and tails becomes obvious, and the numerical values of the chances are found by putting $H=T=\frac{1}{2}$.

M. de Montessus' book, though it does not possess the literary charm of Bertrand's, or the critical subtlety of Poincaré's, is brightly written and gives an interesting sketch of a considerable number of applications in varied fields.

Unfortunately, the conciseness demanded by the size of the book has rendered the argument occasionally obscure, an obscurity which is increased by misprints and slips, however trivial, in the analysis; for instance, at pages 74, 97, 104, 123, 130, 154, 180.

M. de Montessus gives some interesting calculations as to the probable distribution of honours at whist, but the assumption that holding four honours and getting the odd trick are independent events seems to require consideration.

Some useful bibliographical references are appended to each chapter. Professor Karl Pearson's researches on the generalised theory of error are not, however, mentioned; an omission of which the *Encyclopädie der mathematischen Wissenschaften* is equally guilty.

An interesting chapter is devoted to the probabilities of artillery fire, a subject which has received much attention on the continent. For instance: "How many shots will be required to give an even chance of four hits on a given target, assuming given probable errors in the shooting." The author deals with a similar problem in §133, but unfortunately the problem is incompletely stated, and the solution given is unsatisfactory as it stands. C. S. JACKSON.

**Über das Wesen der Mathematik.** Rede gehalten am 11 März 1908 in der öffentlichen Sitzung der k. bayerischen Akademie der Wissenschaften. Von A. Voss. Erweitert und mit Anmerkungen versehen. Pp. 98. Preis geheftet M. 3.60. (Leipzig und Berlin: B. G. Teubner, 1908.)

This is an excellent address. It is the object of such academical addresses both to stimulate the interest of those whose life is not bound up with the subjects treated, and to open up more general views to those whose time is spent in special investigations. Both objects are laudable; the latter because sometimes a person's work suffers from ignorance of what is being done in other—perhaps neighbouring—domains; the former because candid people admit that narrow-mindedness about, say, mathematics, even on the part of non-mathematicians, is not advisable. Both objects seem very satisfactorily attained here.

In so far as it is mathematics which enables us to describe, for our practical or intellectual needs, the processes of nature, mathematics is one of the foundations of our present civilization. And yet mathematics, the oldest work of the human mind, interests the majority of people the least of all sciences because they do not see its importance. This is partly owing to an antiquated method pursued in some schools in the teaching of mathematics (pp. 5-6, cf. pp. 94-97), and partly owing to the difficulty of answering, in a form that can be generally understood, the question (p. 7): In what consists really the essence (Wesen) of mathematics? By what means is it brought about that we have here the only science whose materials and results are apodictic in form, and what, in particular, has the last century done in the development of mathematics in this direction of the theory of knowledge?

Prof. Voss first sketches (pp. 7-22) the history of mathematics from the Rhind Papyrus, of about 1800 B.C., to Laplace. The lengthy historical and critical notes and the references are very valuable; that on kinematics (p. 15) and the critique of Laplace's idea of a "world formula" (p. 21) seem especially instructive. Then (pp. 22-70) an account of what the nineteenth century has done in the principles of mathematics, and in the perfecting of the conceptions of variable, function, continuity, and irrational numbers, and in the formation and application of the theory of aggregates, is given. Then (pp. 71-87) is sketched the influence which the development of pure mathematics has had on applied, especially (almost exclusively) on Geometry. The rest of the address is occupied chiefly with the questions of the relations of mathematics to logic and to intuition.

Four criticisms may, it seems to me, be made. The first may be merely a matter of personal opinion. It has always seemed to me that the great gap between school-mathematics and mathematics proper may be nearly filled by the idea, which we usually meet with first in Cartesian geometry, of *one* algebraical equation between $x$ and $y$ serving to express the relations between an *infinity* of abscissae and ordinates, and expressing, for example, a whole geometrical curve or a whole kinematical motion. This most beautiful idea, after the acquisition of which the infinitesimal calculus and the concepts of variable and function seem to offer no great difficulties, might be emphasised more strongly on p. 14. Secondly, the part about Leibniz (pp. 19-20) might, with advantage, be expanded. Thirdly, there is the formalist confusion of number with sign (pp. 25-26, 32, 42, 44), against which I have protested in this *Gazette*,* though here, indeed, the sign is not a mere sign, but a sign for the ordering activities of our understanding.† Fourthly, the notes on the relations of mathematics to logic, and the questions of principle about the redefinition of equality, the "principle of permanence," the existence of irrationals, Schoenflies' "solution" of the paradoxes in the theory of aggregates, "finite definition" and Hibbert's "definitions" by postulates (pp. 24-26, 88-92, 34, 37, 62, 63, 87-88) are not so thorough as the rest of the work. And thus the statement that the essence of mathematics lies in the doctrine of numbers, and that the characteristic of mathematics is symbolism (pp. 27, 26), seems to be only part of the truth.‡

We must, I think, regard this address as a sign of the good influence that the *Encyklopädie der mathematischen Wissenschaften* is exercising, by its emphasis on the facts that mathematics has a history, that it is growing at the present time, and that these facts are important in education. It does not wholly lose sight even of the logical aspect of mathematics. PHILIP E. B. JOURDAIN.

**Gruppen- und Substitutionentheorie.** By Dr. EUGEN NETTO. Price M. 5.20. (Leipzig: Göschen, 1908. Sammlung Schubert LV.)

A few years back a reviewer pointed out in these columns§ the need of a really elementary Introduction to the Theory of Groups. Attempts have since been

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*Jan. 1908, pp. 207-208 (on p. 207, l. 22, *for* temporary *read* temporal; l. 31, *after* it *add*, or rather its sign; l. 41, *for* geometrical *read* kinematical); July, 1908, p. 307.

†So that *cardinal* numbers would seem to be excluded.

‡Two things may lead to confusion: Numbers are spoken of on p. 28 as being made continuously variable; and, on p. 64, $\omega$ is said to be a *power*, an ordinal is said to be a cardinal.

§No. 50, Vol. III., March 1905, pp. 165, 166.

made to satisfy this need, among which is included Dr. Netto's treatise. The author has succeeded in producing a book which is elementary and is quite the sort of thing a beginner wants. The exposition is very clear, and the earlier chapters contain many useful illustrative examples. These unfortunately grow scarcer as the book proceeds, and examples are nowhere provided for the student to test his own skill.

The scope of the treatise includes the theory of composition—series, chief-series, Sylow subgroups, and Abelian groups, but not the theory of commutants or groups of isomorphisms. A considerable amount of space is devoted to soluble groups. It is a pity that the list of results on p. 122 is inserted, since many of them have been rendered obsolete by Burnside's proof that every group of order $p^{\alpha} q^{\beta}$ is soluble—a result not included in Dr. Netto's list.

Considering the elementary nature of the book, the treatment of permutation-groups in chaps. 9, 10, 11 is very thorough; and will prove useful even to more advanced students.

To stimulate interest the author quotes many results of which proofs are omitted for lack of space. The idea is a good one, but its value is largely neutralised by the fact that no references are given to places where proofs may be found. This seems a mistake, even if it is freely admitted that references in the case of *proved* results are out of place in an elementary treatise.

The book, though good as far as it goes, needs supplementing, for it is impossible nowadays to study group-theory satisfactorily without some acquaintance with the substitution-groups which promise to play a more important part even than permutation-groups in future developments of the subject.

HAROLD HILTON.

**Obras sobre Mathematica.** By Dr. F. GOMES TEIXEIRA. Published by the order of the Portuguese Government. Vols. I. and IV. (University of Coimbra.) Pp. 391, 402. 1904 and 1908.

The major part of these handsomely printed quartos is concerned with the developments of functions of various kinds in series. For instance, the first hundred pages of Volume i. comprise a series of papers on Taylor's Theorem in the case of functions of both real and complex variables, first in an elementary treatment and then by the methods of Cauchy, Riemann, Weierstrass and Mittag-Leffler. The paper is completed by a discussion of the series of Burmann and Lagrange with a generalisation of the former. The memoir on curves having the same evolute as the ellipse,* was crowned in 1898 by the University of Belgium. Memoirs on the convergence of certain formulae of interpolation, and articles on infinitesimal analysis form an important part of the volume, which also contains matter of a more elementary kind in connection with plane analytical geometry. Vol. iv. is a revised and enlarged edition of a work entitled *Tratado de las curvas especiales notables*, crowned by the University of Madrid in 1899. It apparently owed its existence to a suggestion thrown out by M. Haton de la Goupillière in the early days of the *Intermédiaire*. It is now translated into French. Another volume will be necessary to complete this great undertaking, and any reader who has the whole of Dr. Texeira's work in his hands, as well as the classic treatise of Gino Loria, will have at his disposal the results of an enormous amount of research. For each curve that has a distinctive name † he discusses the form, methods of construction, areas, rectification, properties and history, the relations of curves one with another, and the various problems of practical and theoretical interest into which the curves enter. The bibliographical references are full, and the author claims that his labours embody results that are not to be found elsewhere. The list of curves contains about 120 names. Cubics are dismissed in 150 pages, and quartics in about as many. Some fifty pages are devoted to curves of the sixth and eighth degrees. We have noticed a few misprints, such as "treatrise," "Clebesch," etc., which no doubt will

* Toroids.

† How necessary such work is may be inferred from a passage in Vol. I. "These curves have been investigated by Petersen, who calls them 'curves of constant power'; by Humbert, who names them 'cyclic curves'; and by d'Ocagne, in whose hands they become 'isotropic curves.'"

disappear in a second edition. The next volume will deal with equations, which according to the parameter involved represent either algebraical or transcendental curves—parabolas and hyperbolas of any order, cycloidal curves, and the like. Two chapters will be devoted to gauche curves, and one to the theory of Poinsot's polhode and herpolhode. The book will be a perfect storehouse of material for examination questions on the applications of the calculus to analytical geometry.

**Œuvres de Charles Hermite, publiées sous les Auspices de l'Académie des Sciences.** Par EMILE PICARD. Vol. II. Pp. 520. (Gauthier-Villars, 1908.)

As in the first volume, M. Picard has arranged the contents in chronological order. The memoirs date from 1858 to 1872. In 1856 Hermite had been admitted to the Institute, and six years later a chair was created for him at the Ecole Normale. As we all know he was afterwards appointed through the influence of Pasteur to the Ecole Polytechnique, and eventually, in 1869, succeeded Duhamel as Professor of Analysis at the Sorbonne. The memoirs in the present volume are the outcome of his labours during the years when perhaps his fertility was greatest. Fifteen years before, he had written to Jacobi, a letter of a few pages, which placed him, then barely twenty years of age, among the finest analysts of Europe. The letter, it may be added, dealt with a question in connection with hyper-elliptical functions. His special interests for the next twenty years lay in the domain of pure number and, in spite of the inconveniences attending physical frailty, his power of invention was now at its highest. He took his place with Sylvester and Cayley in the creation of the theory of algebraical forms, he wrote his great memoir on the equation of the fifth degree, and he discovered the properties of the modular function, and the nature of modular equations, with their application to the theory of elliptic functions. M. Bourget has borne the lion's share in the editing of the memoir on the equation of the fifth degree for the present volume, having fully revised and checked the whole of the calculations. Many other questions of absorbing interest are dealt with in the present volume, and are striking enough if merely as illustrating the width of Hermite's attainment and the elegance of his methods. We hope that as soon as the last of these volumes is published, this worthy monument to one of the greatest names on the roll of French mathematicians will be crowned by at least a selection from his correspondence. For in the letters he wrote and received we would see adequately reflected the mathematical life of Europe almost from the times of Gauss, Cauchy, Jacobi and Dirichlet to that fatal day when stern Death laid his icy finger on one of the best and purest of men. We had almost forgotten to add that an additional interest attaches to Vol. iv., in that it contains an excellent portrait of Hermite at the age of fifty or thereabouts.

**Vorlesungen über Geschichte der Mathematik.** By MORITZ CANTOR. Vol. I. Third Edition. Pp. vi, 941. (From the earliest times to 1200 B.C.) Vol. II. Pp. vi, 1113. (1759-1799.)

It is fifteen years since the second edition of the first volume of Cantor's *Vorlesungen* made its appearance, and last year a third edition was called for. After the completion of Vol. iii., which brought this monumental work up to 1758, the author felt that the time had come when the claims of advancing years were strong enough to force him to leave the completion of his *magnum opus* to others. He must have felt that he had not lived in vain and that an historical school worthy of German traditions had grown up around him, when, inspired by his example and enthusiasm, men were found able and eager to continue his great work under his direction. We are unfortunate in that the health of the aged savant has prevented him from giving that full personal supervision which is so essential in securing a general unity of treatment when the mathematical labours of half a century are divided among so many hands, however individually competent they may be. And when we say that no less than nine monographs by nine men constitute this fourth volume, it is not surprising to find, that although there is less diversity of treatment than might

be expected, yet the volume is not what it would have been had Cantor been able to continue his colossal task. As it is, he had intended to append to this portion of the work a general treatment of the progress of mathematics during the latter part of the eighteenth century. We are, however, disappointed at the absence of what would of course have been a masterly survey of the development of ideas in that period, and greatly regret that the plan had to be abandoned. The greatest mathematical historiographer found himself compelled to be content with a chronological index. Even so, it is one of the most useful features in the book. His *Überblick* gives a mass of information in a couple of dozen pages, fortified with ample cross references to the sections of this volume. Dr. Günther was entrusted with the purely historical section. Under this heading we have accounts of works dealing with the history of the subject, monographs on its various branches, dictionaries, and editions of the works of classical authors. Arithmetic, the Theory of Equations, and the Theory of Numbers fell to the lot of Fl. Cajori. Netto of Giessen writes on Permutations and Combinations, the Theory of Probability, and on Imaginary Series. Elementary Geometry, Mensuration, etc., and the doctrine of parallels as it stood in this period are undertaken by Bobynin. The additions to Trigonometry made by Euler and his contemporaries, with what is called Polygonometry, Polyedrometry and Cyclometry are adequately dealt with by Braunmühl. Analytical Geometry of the Plane and Space is admirably handled by Kommerell, while Gino Loria is quite at home in his treatment of Perspective and the golden age of Descriptive Geometry. The calculus forms, as might be expected, one of the longest of the monographs, forming about one-fifth of the book. This was Vivanti's share in the work, while Differential Equations and the Calculus of Variations fell to C. R. Wallner. When we remember that we have now got to the days when most of the Bernoullis, Bezout, Condorcet, d'Alembert, Arbogast, Euler, Lagrange, Lambert, Laplace, Carnot, Legendre, Monge, etc., were alive, it may be surmised that the mathematician will find this volume as full of good things as the proverbial egg is of meat. And so it is.

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## ERRATA.

P. 323, l. 3 up, should read—"The correspondence of single operands to the sets of operands or of sets . . ."

P. 325. The last sentence should read—"By a cube root of one ought strictly to be meant *any* treposition, and not *only* such an expression as $\frac{-1+\sqrt{-3}}{2}$."

P. 326. In paragraphs 3, 4, 5, for $ux$ read $ux^p$.

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## BOOKS, ETC., RECEIVED.

*The Progress Report of the Indian Mathematical Club.* No. 7. Oct. 15. 1908.

*On the Cardioide.* R. P. PARANJPYE. *The Nine-Point Circle.* M. T. NARANIENGAR. *Questions and Solutions.*

*American Journal of Mathematics.* Edited by F. MORLEY. Vol. XXX. No. 4. Oct. 1908.

*On a Group of Transformations which occurs in the Problem of Several Bodies.* E. O. LOVETT. *Normal Curves of Genus 6, and their Groups of Birational Transformation.* V. SNYDER. *On the Range of Birational Transformation of Curves of Genus greater than the Canonical Form.* V. SNYDER. *A Set of Assumptions for Projection Geometry.* O. VEBLEN and J. W. YOUNG. *On the Pantastroid.* R. P. STEPHENS. *A Table of the Values of $m$ corresponding to given Values of Euler's $\phi$-function of $m$ up to $\phi(m)=1000$.*

*Die Elemente der Mathematik.* By E. BOREL. Translated by P. STÄCKEL. Vol. I. *Arithmetik und Algebra.* Pp. xvi, 431. 8 m. 60. 1908. (Teubner, Leipzig.)

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